Calculation of the Deep Inelastic Scattering Cross Section

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The goal of this paper is to derive the cross section for the inclusive Deep-Inelastic Scattering (DIS) interaction, approximated as a one photon exchange. DIS refers to the interaction in which a lepton scatters off of a nucleon and only the outgoing lepton is measured. In this work, we consider an unpolarized nucleon and lepton.

Looking at Figure 1, the squared amplitude of the reaction

$$e\left(k\right)+N\left(P\right)\rightarrow e\left(k'\right)+X\left(P_X\right)$$

must be found. There is also a convenient factorization possible, as the top half can be considered an electron-quark scattering event, while the bottom half is a quark-quark correlator of the nucleon (see Figure 2).

Using the Feynman Rules for the amplitude $|\mathcal{M}|^2$,

$$\mathcal{M}=\left(\bar{u}\left(k',s_3\right)\left(ie\gamma^{\mu}\right)\left(u\left(k,s_1\right)\right)\left(\frac{-ig_{\mu\nu}}{q^2}\right)\left(\bar{u}(p',s_4)\left(ie_a e\gamma^{\nu}\right)\langle P|\,\bar{\psi}(0)\,|X\rangle\right),$$

where $p'=p+q$ and $k'=k-q$. The spinors u and $\bar{u}$ are for the incoming and outgoing fermion lines respectively, and the quark field in the nucleon is represented by ψ.

Carrying out the metric tensor contraction and multiplying by the complex conjugate,

$$|\mathcal{M}|^2=\left[\frac{-i^3e_ae^2}{q^2}\left(\bar{u}\left(k',s_3\right)\gamma^{\mu}u\left(k,s_1\right)\right)\left(\bar{u}\left(p',s_4\right)\gamma_{\mu}\langle P|\,\bar{\psi}(0)\,|X\rangle\right)\right]$$
$$\times\left[\frac{i^3e_ae^2}{q^2}\left(\bar{u}\left(k',s_3\right)\gamma^{\nu}u\left(k,s_1\right)\right)^*\left(\bar{u}\left(p',s_4\right)\gamma_{\nu}\langle P|\,\bar{\psi}(0)\,|X\rangle\right)^*\right].$$

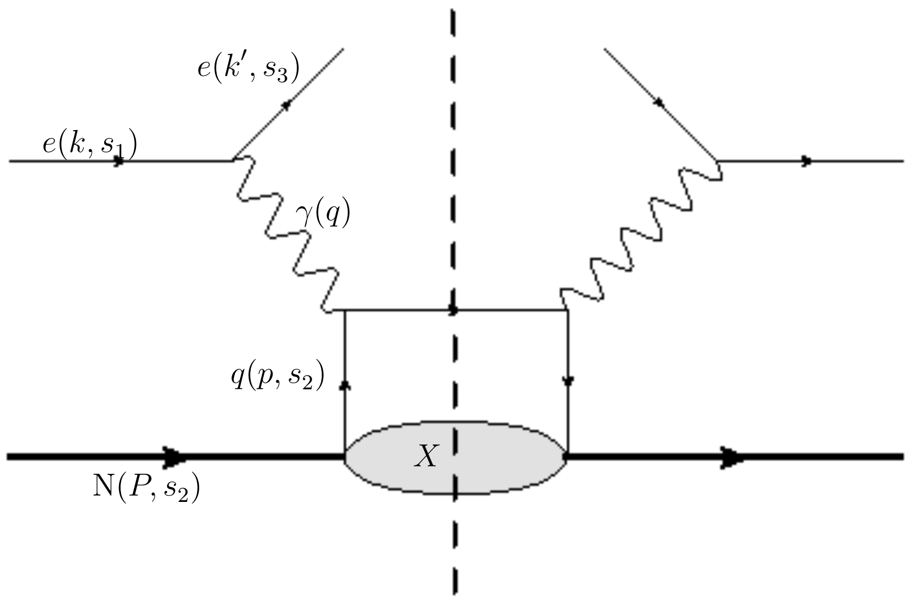

Figure 1: This is the Feynman diagram for the Deep Inelastic Scattering interaction. Each line on the left is labeled by momentum and spin.

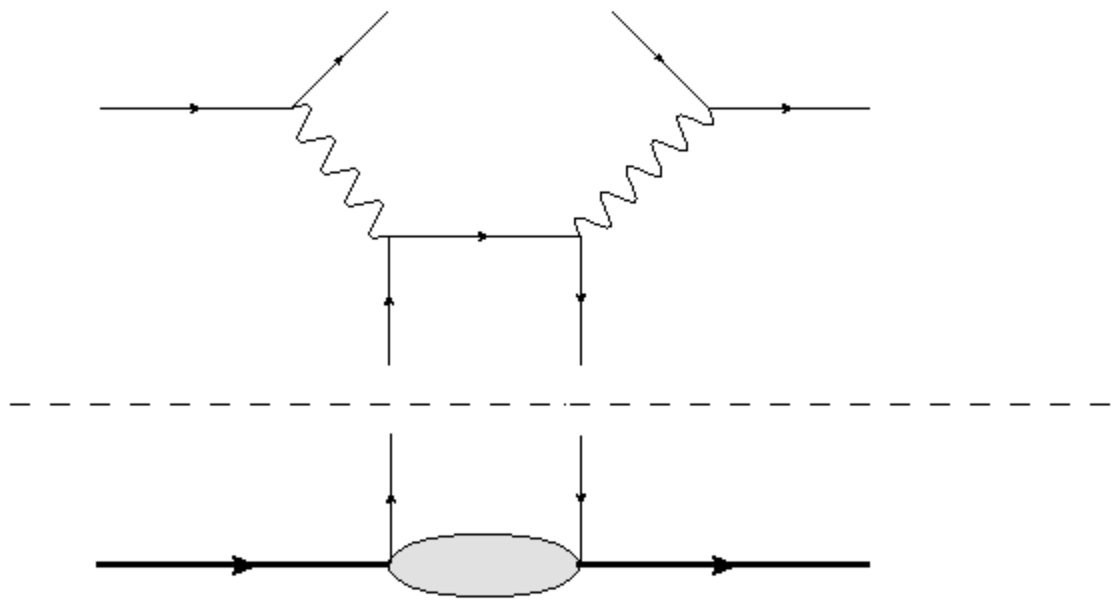

Figure 2: This is the factored Feynman diagram for the DIS interaction.

Collecting terms, using commutative property of complex numbers,

$$|\mathcal{M}|^2 = \frac{e_a^2\, e^4}{q^4} \left[\bar{u}\left(k', s_3\right)\gamma^\mu u\left(k, s_1\right)\right] \left[\bar{u}\left(k', s_3\right)\gamma^\nu u\left(k, s_1\right)\right]$$
$$\times \left[\bar{u}\left(p', s_4\right)\gamma_\mu \langle P|\,\bar{\psi}(0)\,|X\rangle\right] \left[\bar{u}\left(p', s_4\right)\gamma_\nu \langle P|\,\bar{\psi}(0)\,|X\rangle\right]^* .$$

Using Casimir's Trick to average over all the spins and recognizing $\langle P|\,\bar{\psi}(0)\,|X\rangle^* = \langle X|\,\psi(0)\,|P\rangle$,

$$\langle|\mathcal{M}|^2\rangle \;=\; \frac{e_a^2\, e^4}{4q^4} \mathrm{Tr}\left[\gamma^\mu(\not{k}')\gamma^\nu(\not{k})\right] \mathrm{Tr}\left[\gamma_\mu\left(\not{p}'\right)\gamma_\nu \langle P|\,\bar{\psi}(0)\,|X\rangle \langle X|\,\psi(0)\,|P\rangle\right], \tag{1}$$

where $\langle|\mathcal{M}|^2\rangle \equiv \frac{1}{4}\sum\limits_{\text{spins}} |\mathcal{M}|^2$.

The expression for the cross-section in terms of $\langle|\mathcal{M}|^2\rangle$ is

$$\mathrm{d}\sigma = \sum\!\!\!\!\!\!\int_X \frac{\mathrm{d}^3\mathbf{P}_X}{(2\pi)^3\, 2P_X^0} \frac{\mathrm{d}^3\mathbf{k}'}{(2\pi)^3\, 2k'^0} \frac{\mathrm{d}^3\mathbf{p}'}{(2\pi)^3\, 2p'^0} \frac{1}{P\cdot k} \left(\langle|\mathcal{M}|^2\rangle\right)$$
$$\times (2\pi)^4\, \delta^{(4)}\left(p' + P_X + k' - P - k\right). \tag{2}$$

Substituting (1)into (2)and bringing the $\mathbf{k}'$ differential to the other side,

$$\frac{k'^0 \mathrm{d}\sigma}{\mathrm{d}^3\mathbf{k}'} = \frac{1}{4P\cdot k}\frac{1}{8\pi^3}\frac{1}{2}\sum\!\!\!\!\!\!\int_X \frac{\mathrm{d}^3\mathbf{P}_X}{(2\pi)^3\, 2P_X^0}\frac{\mathrm{d}^3\mathbf{p}'}{(2\pi)^3\, 2p'^0}\frac{e_a^2\, e^4}{4q^4}\mathrm{Tr}\left[\gamma^\mu (\not k')\gamma^\nu (\not k)\right]$$
$$\times \mathrm{Tr}\left[\gamma_\mu \left(\not p'\right)\gamma_\nu \left\langle P\right|\bar\psi(0)\left|X\right\rangle \left\langle X\right|\psi(0)\left|P\right\rangle\right](2\pi)^4\,\delta^{(4)}\left(p' + P_X + k' - P - k\right).$$

Recognizing $q = k - k'$, and shifting $\psi(0)$ to $\psi(\xi)$,

$$\frac{k'^0 \mathrm{d}\sigma}{\mathrm{d}^3\mathbf{k}'} = \frac{1}{4P\cdot k}\frac{1}{8\pi^3}\frac{1}{2}\sum\!\!\!\!\!\!\int_X \frac{\mathrm{d}^3\mathbf{P}_X}{(2\pi)^3\, 2P_X^0}\frac{\mathrm{d}^3\mathbf{p}'}{(2\pi)^3\, 2p'^0}\frac{e_a^2\, e^4}{4q^4}\mathrm{Tr}\left[\gamma^\mu (\not k')\gamma^\nu (\not k)\right]$$
$$\times\, \mathrm{Tr}\left[\gamma_\mu \left(\not p'\right)\gamma_\nu \left\langle P\right|\bar\psi(0)\left|X\right\rangle \left\langle X\right| e^{-i\hat P\cdot\xi}\psi(\xi)\, e^{i\hat P\cdot \xi}\left|P\right\rangle\right]$$
$$\times\,(2\pi)^4\,\delta^{(4)}\left(p' + P_X - P - q\right).$$

Applying the momentum operators,

$$\frac{k'^0 \mathrm{d}\sigma}{\mathrm{d}^3\mathbf{k}'} = \frac{1}{4P\cdot k}\frac{1}{8\pi^3}\frac{1}{2}\sum\!\!\!\!\!\!\int_X \frac{\mathrm{d}^3\mathbf{P}_X}{(2\pi)^3\, 2P_X^0}\frac{\mathrm{d}^3\mathbf{p}'}{(2\pi)^3\, 2p'^0}\frac{e_a^2\, e^4}{4q^4}\mathrm{Tr}\left[\gamma^\mu (\not k')\gamma^\nu (\not k)\right]$$
$$\times\, \mathrm{Tr}\left[\gamma_\mu \left(\not p'\right)\gamma_\nu \left\langle P\right|\bar\psi(0)\left|X\right\rangle \left\langle X\right| e^{-iP_X\cdot\xi}\psi(\xi)\, e^{iP\cdot \xi}\left|P\right\rangle\right]$$
$$\times\,(2\pi)^4\,\delta^{(4)}\left(p' + P_X - P - q\right),$$

recognizing the completeness relation in X, and pulling the exponentials out of the trace (as they are complex numbers) gives

$$\frac{k'^0 \mathrm{d}\sigma}{\mathrm{d}^3\mathbf{k}'} = \frac{1}{4P\cdot k}\frac{1}{8\pi^3}\frac{1}{2}\int \frac{\mathrm{d}^3\mathbf{p}'}{(2\pi)^3\, 2p'^0}\frac{e_a^2\, e^4}{4q^4}\mathrm{Tr}\left[\gamma^\mu (\not k')\gamma^\nu (\not k)\right]\mathrm{Tr}\left[\gamma_\mu \left(\not p'\right)\gamma_\nu \left\langle P\right|\bar\psi(0)\,\psi(\xi)\left|P\right\rangle\right]$$
$$\times\, e^{-iP_X\cdot\xi}e^{iP\cdot\xi}\,(2\pi)^4\,\delta^{(4)}\left(p' + P_X - P - q\right).$$

Using the definition

$$(2\pi)^n\,\delta^n(\lambda) = \int \mathrm{d}^n\phi\, e^{i\lambda\cdot\phi} \tag{3}$$

and the relation

$$\int \frac{\mathrm{d}^3\mathbf{p}'}{(2\pi)^3\, 2p'^0} = \int \frac{\mathrm{d}^4 p'}{(2\pi)^3}\delta\left(p'^2\right),$$

and pulling $\frac{1}{2\pi}$ into the integration to the match order of the differential,

$$\frac{k'^0 \mathrm{d}\sigma}{\mathrm{d}^3\mathbf{k}'} = \frac{1}{4P\cdot k}\frac{1}{4\pi^2}\frac{1}{2}\int\frac{\mathrm{d}^4p'}{(2\pi)^4}\mathrm{d}^4\xi\frac{e_a^2\,e^4}{4q^4}\mathrm{Tr}\left[\gamma^\mu(\not{k}')\gamma^\nu(\not{k})\right]\mathrm{Tr}\left[\gamma_\mu\left(\not{p}'\right)\gamma_\nu\left\langle P\right|\bar{\psi}\left(0\right)\psi\left(\xi\right)\left|P\right\rangle\right] \times e^{-iP_X\cdot\xi}e^{iP\cdot\xi}e^{i\left(p'+P_X-P-q\right)\cdot\xi}\delta\left(p'^2\right).$$

Simplifying the exponential and substituting $p' = p+q$ in the delta function, exponential, and differential ($\mathrm{d}p' = \mathrm{d}p$),

$$\frac{k'^0 \mathrm{d}\sigma}{\mathrm{d}^3\mathbf{k}'} = \frac{1}{4P\cdot k}\frac{1}{4\pi^2}\frac{1}{2}\frac{e_a^2\,e^4}{4q^4}\int\frac{\mathrm{d}^4p}{(2\pi)^4}\mathrm{d}^4\xi\mathrm{Tr}\left[\gamma^\mu(\not{k}')\gamma^\nu(\not{k})\right]\mathrm{Tr}\left[\gamma_\mu\left(\not{p}'\right)\gamma_\nu\left\langle P\right|\bar{\psi}\left(0\right)\psi\left(\xi\right)\left|P\right\rangle\right] \times e^{ip\cdot\xi}\delta\left((p+q)^2\right).$$

Expanding $(p+q)^2 = -Q^2+2p\cdot q$, where $q^2 = -Q^2$ and using the plus/minus basis gives $-Q^2+2p\cdot q = -Q^2+2p^+q^-$. Since $p^+ = xP^+$, where x denotes the fraction of the momentum the quark carries from the nucleon, $-Q^2+2p^+q^- = -Q^2+2xP^+q^-$. So $(p+q)^2 = -Q^2+2xP^+q^-$. The differential can also be expanded and rewritten in terms of x, $\mathrm{d}^4p = \mathrm{d}p^+\mathrm{d}p^-\mathrm{d}^2\mathbf{p_T} = \mathrm{d}xP^+\mathrm{d}p^-\mathrm{d}^2\mathbf{p_T}$:

$$\frac{k'^0 \mathrm{d}\sigma}{\mathrm{d}^3\mathbf{k}'} = \frac{1}{4P\cdot k}\frac{1}{4\pi^2}\frac{1}{2}\frac{e_a^2\,e^4}{4q^4}\int\frac{\mathrm{d}xP^+\mathrm{d}p^-\mathrm{d}^2\mathbf{p_T}}{(2\pi)^4}\mathrm{d}^4\xi\mathrm{Tr}\left[\gamma^\mu(\not{k}')\gamma^\nu(\not{k})\right] \times \mathrm{Tr}\left[\gamma_\mu\left(\not{p}'\right)\gamma_\nu\left\langle P\right|\bar{\psi}\left(0\right)\psi\left(\xi\right)\left|P\right\rangle\right]e^{ip\cdot\xi}\delta\left(-Q^2+2xP^+q^-\right).$$

Since the Bjorken variable $x_B = \frac{Q^2}{2P^+q^-}$, the delta function can be written as $\frac{1}{2P^+q^-}\delta\left(x-x_B\right)$. Making this substitution and canceling P^+ terms,

$$\frac{k'^0 \mathrm{d}\sigma}{\mathrm{d}^3\mathbf{k}'} = \frac{1}{4P\cdot k}\frac{1}{4\pi^2}\frac{1}{2}\frac{e_a^2\,e^4}{4q^4}\int\frac{\mathrm{d}x\mathrm{d}p^-\mathrm{d}^2\mathbf{p_T}}{(2\pi)^4}\mathrm{d}^4\xi\mathrm{Tr}\left[\gamma^\mu(\not{k}')\gamma^\nu(\not{k})\right] \times \mathrm{Tr}\left[\gamma_\mu\left(\not{p}'\right)\gamma_\nu\left\langle P\right|\bar{\psi}\left(0\right)\psi\left(\xi\right)\left|P\right\rangle\right]e^{ip\cdot\xi}\frac{1}{2q^-}\delta\left(x-x_B\right).$$

Integrating the x dependence then expanding ξ in the plus/minus base,

$$\frac{k'^0 \mathrm{d}\sigma}{\mathrm{d}^3\mathbf{k}'} = \frac{1}{4P\cdot k}\frac{1}{4\pi^2}\frac{1}{2}\frac{e_a^2\,e^4}{4q^4}\int\frac{\mathrm{d}p^-\mathrm{d}^2\mathbf{p_T}}{(2\pi)^4}\mathrm{d}\xi^+\mathrm{d}\xi^-\mathrm{d}^2\xi_\mathbf{T}\mathrm{Tr}\left[\gamma^\mu(\not{k}')\gamma^\nu(\not{k})\right] \times \mathrm{Tr}\left[\gamma_\mu\left(\not{p}'\right)\gamma_\nu\left\langle P\right|\bar{\psi}\left(0\right)\psi\left(\xi^+,\xi^-,\xi_\mathbf{T}\right)\left|P\right\rangle\right]e^{i\left(p^-\xi^++p^+\xi^--\mathbf{p_T}\cdot\xi_\mathbf{T}\right)}\frac{1}{2q^-}.$$

After the integration over x, the delta function forces all $x \to x_B$. For simplicity, the subscript B will be dropped and x denotes x_B through the rest of the derivation. Using (3), $\frac{\mathrm{d}p^-}{2\pi} e^{ip^-\xi^+} = \delta\left(\xi^+\right)$ and $\frac{\mathrm{d}^2\mathbf{p_T}}{(2\pi)^2} e^{-i\mathbf{p_T}\cdot\xi_\mathbf{T}} = \delta\left(-\xi_\mathbf{T}\right)$,

$$\frac{k'^0 \mathrm{d}\sigma}{\mathrm{d}^3\mathbf{k}'} = \frac{1}{4P\cdot k}\frac{1}{4\pi^2}\frac{1}{2}\frac{e_a^2\, e^4}{4q^4}\int \mathrm{d}\xi^+ \mathrm{d}\xi^- \mathrm{d}^2\xi_\mathbf{T} \mathrm{Tr}\left[\gamma^\mu (\not{k}')\gamma^\nu (\not{k})\right]$$
$$\times \mathrm{Tr}\left[\gamma_\mu \left(\not{p}'\right)\gamma_\nu \langle P|\, \bar{\psi}(0)\,\psi\left(\xi^+,\xi^-,\xi_\mathbf{T}\right)|P\rangle\right] e^{i\left(p^+\xi^-\right)}\delta\left(\xi^+\right)\delta\left(-\xi_\mathbf{T}\right)\frac{1}{2q^-}\frac{1}{(2\pi)}.$$

Integrating over all the delta functions and expanding the second trace in terms of indices,

$$\frac{k'^0 \mathrm{d}\sigma}{\mathrm{d}^3\mathbf{k}'} = \frac{1}{4P\cdot k}\frac{1}{4\pi^2}\frac{1}{2}\frac{e_a^2\, e^4}{4q^4}\int \frac{\mathrm{d}\xi^-}{(2\pi)} \mathrm{Tr}\left[\gamma^\mu (\not{k}')\gamma^\nu (\not{k})\right]$$
$$\times \left[\gamma_{\mu_{ij}}\left(\not{p}'\right)_{jk}\gamma_{\nu_{k\ell}} \langle P|\, \bar{\psi}_\ell(0)\,\psi_i\left(\xi^-\right)|P\rangle\right] e^{i\left(p^+\xi^-\right)}\frac{1}{2q^-}$$

The correlator $\Phi(x)$ is defined as

$$\Phi_{\ell i}(x) = \frac{1}{2}\int \frac{\mathrm{d}\xi^-}{2\pi} e^{i\left(p^+\xi^-\right)} \langle P|\, \bar{\psi}_i(0)\,\psi_\ell\left(\xi^-\right)|P\rangle\,,$$

so substituting this in,

$$\frac{k'^0 \mathrm{d}\sigma}{\mathrm{d}^3\mathbf{k}'} = \frac{1}{4P\cdot k}\frac{1}{4\pi^2}\frac{e_a^2\, e^4}{4q^4}\frac{1}{2q^-}\mathrm{Tr}\left[\gamma^\mu (\not{k}')\gamma^\nu (\not{k})\right]\left[\gamma_{\mu_{ij}}\left(\not{p}'\right)_{jk}\gamma_{\nu_{k\ell}}\Phi_{\ell i}(x)\right].$$

Reintroducing the trace,

$$\frac{k'^0 \mathrm{d}\sigma}{\mathrm{d}^3\mathbf{k}'} = \frac{1}{4P\cdot k}\frac{1}{4\pi^2}\frac{e_a^2\, e^4}{4q^4}\frac{1}{2q^-}\mathrm{Tr}\left[\gamma^\mu (\not{k}')\gamma^\nu (\not{k})\right]\mathrm{Tr}\left[\gamma_\mu\left(\not{p}'\right)\gamma_\nu\Phi(x)\right].$$

Expanding $\Phi(x)$ (a 4 by 4 matrix) in the a basis of gamma matrices (all 4 by 4 matrices),

$$\Phi(x) \approx \frac{1}{4}\left(\mathrm{Tr}[\Phi\gamma^+]\gamma^- - \mathrm{Tr}[\Phi\gamma^+\gamma^5]\gamma^-\gamma^5 + \mathrm{Tr}[\Phi i\sigma^{i+}\gamma^5]i\sigma^{i-}\gamma^5\right).$$

Only the first term of the trace is considered, as the cross-section is for the unpolarized case, making the second term 0, and the third term is 0 when inserted into the trace of the cross-section. The trace of the first term gives twice the PDF for a given quark flavor (as the nucleon spins were already averaged over earlier). So,

$$\frac{k'^0 \mathrm{d}\sigma}{\mathrm{d}^3\mathbf{k}'} = \frac{1}{4P\cdot k}\frac{1}{4\pi^2}\frac{e_a^2\, e^4}{4q^4}\frac{1}{2q^-}\mathrm{Tr}\left[\gamma^\mu(\not{k}')\gamma^\nu(\not{k})\right]\frac{1}{4}\left(2f_1\left(x\right)\right)\mathrm{Tr}\left[\gamma_\mu\left(\not{p}'\right)\gamma_\nu\gamma^-\right].$$

Multiplying by a factor of p^+/p^+ inside the second trace,

$$\frac{k'^0 \mathrm{d}\sigma}{\mathrm{d}^3\mathbf{k}'} = \frac{1}{4P\cdot k}\frac{1}{4\pi^2}\frac{e_a^2\, e^4}{4q^4}\frac{1}{2q^-}\mathrm{Tr}\left[\gamma^\mu(\not{k}')\gamma^\nu(\not{k})\right]\frac{1}{4}\left(2f_1\left(x\right)\right)\mathrm{Tr}\left[\gamma_\mu\left(\not{p}'\right)\gamma_\nu\gamma^- p^+/p^+\right].$$

Pulling the $1/p^+$ out of the trace (one component of 4-momentum)and using the relation $\gamma^- p^+ = \not{p}$,

$$\frac{k'^0 \mathrm{d}\sigma}{\mathrm{d}^3\mathbf{k}'} = \frac{1}{4P\cdot k}\frac{1}{4\pi^2}\frac{e_a^2\, e^4}{16q^4}\frac{f_1\left(x\right)}{q^- p^+}\mathrm{Tr}\left[\gamma^\mu(\not{k}')\gamma^\nu(\not{k})\right]\mathrm{Tr}\left[\gamma_\mu\left(\not{p}'\right)\gamma_\nu\not{p}\right].$$

Evaluating the traces, the four momentum can be pulled out of the trace, leaving products of four gamma matrices. So,

$$\frac{k'^0 \mathrm{d}\sigma}{\mathrm{d}^3\mathbf{k}'} = \frac{1}{4P\cdot k}\frac{1}{4\pi^2}\frac{e_a^2\, e^4}{16q^4}\frac{f_1\left(x\right)}{p\cdot q}\left[\left(k'\right)_\epsilon k_\delta \mathrm{Tr}\left[\gamma^\mu\gamma^\epsilon\gamma^\nu\gamma^\delta\right]\right]\left[\left(p'\right)^\alpha p^\sigma \mathrm{Tr}\left[\gamma_\mu\gamma_\alpha\gamma_\nu\gamma_\sigma\right]\right].$$

Using the properties of the trace of gamma matrices,

$$\begin{aligned}\frac{k'^0 \mathrm{d}\sigma}{\mathrm{d}^3\mathbf{k}'} = \frac{1}{4P\cdot k}\frac{1}{4\pi^2}\frac{e_a^2\, e^4}{16q^4}\frac{f_1\left(x\right)}{p\cdot q}&\left[4\left(k'\right)_\epsilon k_\delta\left(g^{\mu\epsilon}g^{\nu\delta} - g^{\mu\nu}g^{\epsilon\delta} + g^{\mu\delta}g^{\epsilon\nu}\right)\right]\\ &\times\left[4\left(p'\right)^\alpha p^\sigma\left(g_{\mu\alpha}g_{\nu\sigma} - g_{\mu\nu}g_{\alpha\sigma} + g_{\mu\sigma}g_{\alpha\nu}\right)\right].\end{aligned}$$

Contracting indices with the metric tensors,

$$\begin{aligned}\frac{k'^0 \mathrm{d}\sigma}{\mathrm{d}^3\mathbf{k}'} = \frac{1}{4P\cdot k}\frac{1}{4\pi^2}\frac{e_a^2\, e^4}{q^4}\frac{f_1\left(x\right)}{p\cdot q}&\left[\left(k'\right)^\mu k^\nu - g^{\mu\nu}\left(k'\cdot k\right) + k^\mu\left(k'\right)^\nu\right]\\ &\times\left[\left(p'\right)_\mu p_\nu - g_{\mu\nu}\left(p'\cdot p\right) + p_\mu\left(p'\right)_\nu\right].\end{aligned}$$

Distributing the 4-momenta,

$$\begin{aligned}\frac{k'^0 \mathrm{d}\sigma}{\mathrm{d}^3\mathbf{k}'} = \frac{1}{4P\cdot k}\frac{1}{4\pi^2}\frac{e_a^2\, e^4}{q^4}\frac{f_1\left(x\right)}{p\cdot q}&[\left(k'\cdot p'\right)\left(k\cdot p\right) - \left(k'\cdot k\right)\left(p'\cdot p\right) + \left(k\cdot p\right)\left(k'\cdot p'\right) - \left(p'\cdot p\right)\left(k'\cdot k\right)\\ &+4\left(p'\cdot p\right)\left(k'\cdot k\right) - \left(p\cdot p'\right)\left(k'\cdot k\right) + \left(k\cdot p'\right)\left(k'\cdot p\right) - \left(k'\cdot k\right)\left(p'\cdot p\right) + \left(k\cdot p\right)\left(p'\cdot k'\right)].\end{aligned}$$

Since the dot product is commutative,

$$\frac{k'^0 \mathrm{d}\sigma}{\mathrm{d}^3\mathbf{k}'} = \frac{1}{4P\cdot k}\frac{1}{4\pi^2}\frac{e_a^2\, e^4}{q^4}\frac{f_1(x)}{p\cdot q}\left[2\left(k'\cdot p'\right)(k\cdot p) + 2\left(k\cdot p'\right)\left(k'\cdot p\right)\right].$$

The Mandelstam variables are

$$\hat{s} = \left(k'-p'\right)^2 = (k-p)^2 = 2k'\cdot p' = 2k\cdot p, \tag{4}$$
$$\hat{t} = \left(k-p'\right)^2 = \left(k'-p\right)^2 = -2k\cdot p' = -2k'\cdot p, \tag{5}$$
$$\hat{u} = \left(k'-k\right)^2 = \left(k-k'\right)^2 = q^2 = -Q^2, \tag{6}$$
$$S = (P+k)^2 = 2P\cdot k = 2P^+k^-. \tag{7}$$

So expressing the cross section in terms of these,

$$\frac{k'^0 \mathrm{d}\sigma}{\mathrm{d}^3\mathbf{k}'} = \frac{1}{4P\cdot k}\frac{e_a^2\, e^4}{4\pi^2}\frac{f_1(x)}{p\cdot q}\left[\frac{\hat{s}^2+\hat{t}^2}{2\hat{u}^2}\right].$$

The remaining scalar products can be written in terms of x and y, which are defined as:

$$x = \frac{Q^2}{2P\cdot q},$$
$$y = \frac{P\cdot q}{P\cdot k}.$$

So, $\frac{1}{4P\cdot k} = \frac{P\cdot q}{4P\cdot q P\cdot k} = \frac{xy}{2Q^2}$. And $p\cdot q = xP\cdot q = x\frac{Q^2}{2x} = \frac{Q^2}{2}$.

$$\frac{k'^0 \mathrm{d}\sigma}{\mathrm{d}^3\mathbf{k}'} = \frac{xy}{Q^4}\frac{f_1(x)\, e_a^2\, e^4}{4\pi^2}\left[\frac{\hat{s}^2+\hat{t}^2}{2\hat{u}^2}\right].$$

Also using $\alpha_{em} = \frac{e^2}{4\pi}$,

$$\frac{k'^0 \mathrm{d}\sigma}{\mathrm{d}^3\mathbf{k}'} = \frac{4xyf_1(x)\, e_a^2\, \alpha_{em}^2}{Q^4}\left[\frac{\hat{s}^2+\hat{t}^2}{2\hat{u}^2}\right]. \tag{8}$$

To convert the Mandelstam variables into x and y,

$$\hat{s} = 2p\cdot k = 2xP\cdot k = 2\frac{Q^2 P\cdot k}{2P\cdot q} = \frac{Q^2}{y},$$

$$\hat{t} = -2k' \cdot p = -2\left(k'\right)^{-} xP^{+} = -2\frac{Q^2}{2P^{+}q^{-}}\left(P^{+}\left(k'\right)^{-}\right) = -\frac{Q^2\left(k'\right)^{-}}{q^{-}}.$$

To get an expression for $\left(k'\right)^{-}$,

$$y = \frac{P \cdot q}{P \cdot k} = \frac{P \cdot \left(k - k'\right)}{P \cdot k} = 1 - \frac{P \cdot k'}{P \cdot k} = 1 - \frac{P^{+}\left(k'\right)^{-}}{P^{+}k^{-}} = 1 - \frac{\left(k'\right)^{-}}{k^{-}}.$$

Therefore, $\left(k'\right)^{-} = k^{-}(1-y)$. From (4), $k^{-} = \frac{S}{2P^{+}}$, so $\left(k'\right)^{-} = \frac{S}{2P^{+}}(1-y)$. So,

$$\hat{t} = -\frac{Q^2 S(1-y)}{2P^{+}q^{-}} = -xS(1-y).$$

Also,

$$xyS = \frac{Q^2}{2P \cdot q}\frac{P \cdot q}{P \cdot k}(2P \cdot k) = Q^2. \tag{9}$$

Then,

$$\begin{aligned}
\frac{\hat{s}^2 + \hat{t}^2}{2\hat{u}^2} &= \frac{\left(\frac{Q^2}{y}\right)^2 + \left(-xS(1-y)\right)^2}{2\left(-Q^2\right)^2} \\
&= \frac{\left(Q^4/y^2\right) + \left(xS - Q^2\right)^2}{2Q^4} \\
&= \frac{Q^4 + y^2\left(xS - Q^2\right)^2}{2Q^4 y^2} \\
&= \frac{Q^4 + \left(xyS - yQ^2\right)^2}{2Q^4 y^2} \\
&= \frac{Q^4 + \left(Q^2(1-y)\right)^2}{2Q^4 y^2} \\
&= \frac{1 + \left(1 - 2y + y^2\right)}{2y^2} \\
&= \frac{2 - 2y + y^2}{2y^2} \\
&= \frac{1-y}{y^2} + \frac{1}{2}.
\end{aligned}$$

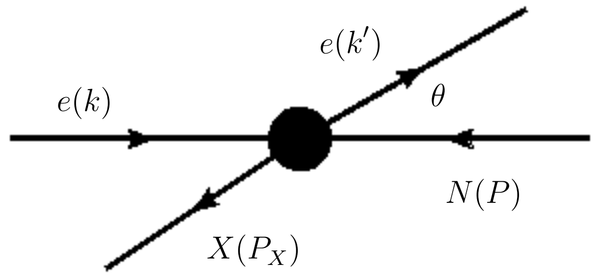

Figure 3: This is the nucleon-lepton interaction, labeled with momenta.

So,

$$\frac{\hat{s}^2+\hat{t}^2}{2\hat{u}^2}=\frac{1-y}{y^2}+\frac{1}{2}. \tag{10}$$

Putting the result of (10)into (8),

$$\frac{k'^0 \mathrm{d}\sigma}{\mathrm{d}^3\mathbf{k}'}=\frac{4xyf_1(x)\,e_a^2\,\alpha_{em}^2}{Q^4}\left[\frac{1-y}{y^2}+\frac{1}{2}\right],$$

and simplifying by distributing gives

$$\frac{k'^0 \mathrm{d}\sigma}{\mathrm{d}^3\mathbf{k}'}=\frac{4f_1(x)\,e_a^2\,\alpha_{em}^2}{Q^4}\left[\frac{x(1-y)}{y}+\frac{xy}{2}\right]. \tag{11}$$

While this is an expression for the cross-section, it is more useful for the LHS to be differential in x and y. First, expressions for Q^2 and y must be found. Consider the nucleon-lepton interaction from the center of mass frame (see Figure 3). Then,

$$\begin{aligned} k &= (E_k, 0, 0, E_k) \\ P &= (E_k, 0, 0, -E_k) \\ k' &= (E_{k'}, E_{k'}\sin\theta, 0, E_{k'}\cos\theta) \\ P_X &= \left(E_{P_X}, -E_{P_X}\sin\theta, 0, -E_{P_X}\cos\theta\right) \end{aligned}$$

So, using properties of 4-momenta and dot products,

$$\begin{aligned}
Q^2 &= -\left(k-k'\right)^2 \\
k-k' &= \left(E_k - E_{k'}, -E_{k'}\sin\theta, 0, E_k - E_{k'}\cos\theta\right) \\
\left(k-k'\right)^2 &= \left(E_k - E_{k'}\right)^2 - \left(\left(-E_{k'}\sin\theta\right)^2 + \left(E_k - E_{k'}\cos\theta\right)^2\right) \\
&= E_k^2 + E_{k'}^2 - 2E_kE_{k'} - \left(E_{k'}^2\sin^2\theta + E_k^2 - 2E_kE_{k'}\cos\theta + E_{k'}^2\cos\theta\right) \\
&= 2E_kE_{k'}\left(\cos\theta - 1\right) \\
\therefore Q^2 &= 2E_kE_{k'}\left(1-\cos\theta\right).
\end{aligned}$$

and

$$\begin{aligned}
y &= \frac{P\cdot\left(k-k'\right)}{P\cdot k} \\
&= \frac{E_k\left(E_k - E_{k'}\right) - \left(-E_k\right)\left(E_k - E_{k'}\cos\theta\right)}{E_k\left(E_k\right) - \left(-E_k\right)\left(E_k\right)} \\
&= \frac{E_k - E_{k'} + E_k - E_{k'}\cos\theta}{2E_k} \\
&= \frac{2E_k - E_{k'}\left(1+\cos\theta\right)}{2E_k} \\
y &= 1 - \frac{E_{k'}}{2E_k}\left(1+\cos\theta\right).
\end{aligned}$$

Expressing the differential in spherical coordinates,

$$\begin{aligned}
\frac{E_{k'}\mathrm{d}\sigma}{\mathrm{d}^3\mathbf{k}'} &= \frac{E_{k'}\mathrm{d}\sigma}{\mathrm{d}E_{k'}E_{k'}^2\sin\theta\mathrm{d}\theta\mathrm{d}\phi} \\
&= \frac{E_{k'}\mathrm{d}\sigma}{\mathrm{d}E_{k'}E_{k'}^2\mathrm{d}\left(\cos\theta\right)\mathrm{d}\phi}.
\end{aligned}$$

Using the Jacobian to change variables from $\mathrm{d}E_k'\mathrm{d}\left(cos\theta\right)$ to $\mathrm{d}Q^2\mathrm{d}y$,

$$\begin{aligned}
\frac{\mathrm{d}Q^2}{\mathrm{d}E_{k'}} &= 2E_k\left(1-\cos\theta\right) \\
\frac{\mathrm{d}Q^2}{\mathrm{d}\left(\cos\theta\right)} &= -2E_kE_{k'} \\
\frac{\mathrm{d}y}{\mathrm{d}E_{k'}} &= \frac{-\left(1+\cos\theta\right)}{2E_k} \\
\frac{\mathrm{d}y}{\mathrm{d}\left(\cos\theta\right)} &= -\frac{E_{k'}}{2E_k}.
\end{aligned}$$

So the matrix $\mathbf{J}$ can be defined as

$$\mathbf{J} = \begin{bmatrix} \frac{\mathrm{d}Q^2}{\mathrm{d}E_{k'}} & \frac{\mathrm{d}Q^2}{\mathrm{d}(\cos\theta)} \\ \frac{\mathrm{d}y}{\mathrm{d}E_{k'}} & \frac{\mathrm{d}y}{\mathrm{d}(\cos\theta)} \end{bmatrix} = \begin{bmatrix} 2E_k\left(1-\cos\theta\right) & -2E_kE_{k'} \\ \frac{-(1+\cos\theta)}{2E_k} & -\frac{E_{k'}}{2E_k} \end{bmatrix}.$$

Taking the determinant of $\mathbf{J}$ to find the Jacobian yields,

$$\left|\det\left(\mathbf{J}\right)\right| = \left|2E_k\left(1-\cos\theta\right)\frac{E_{k'}}{2E_k} - 2E_kE_{k'}\frac{-\left(1+\cos\theta\right)}{2E_k}\right| = 2E_{k'}.$$

So (using (9), $\mathrm{d}Q^2 = yS\mathrm{d}x$),

$$\frac{E_{k'}\mathrm{d}\sigma}{\mathrm{d}E_{k'}E_{k'}^2\mathrm{d}\left(\cos\theta\right)\mathrm{d}\phi} = \frac{\left(2E_{k'}\right)E_{k'}\mathrm{d}\sigma}{E_{k'}^2\mathrm{d}Q^2\mathrm{d}y\mathrm{d}\phi} = \frac{2\mathrm{d}\sigma}{yS\mathrm{d}x\mathrm{d}y\mathrm{d}\phi}.$$

Substituting this into (11),

$$\frac{2\mathrm{d}\sigma}{yS\mathrm{d}x\mathrm{d}y\mathrm{d}\phi} = \frac{4f_1\left(x\right)e_a^2\,\alpha_{em}^2}{Q^4}\left[\frac{x\left(1-y\right)}{y} + \frac{xy}{2}\right]$$

Isolating the differential and integrating over ϕ using the azimuthal symmetry of the interaction, and rearranging for simplification,

$$\frac{\mathrm{d}\sigma}{\mathrm{d}x\mathrm{d}y} = \frac{1}{2}S\left(2\pi\right)\frac{4xf_1\left(x\right)e_a^2\,\alpha_{em}^2}{Q^4}\left[\frac{2\left(1-y\right)}{2} + \frac{y^2}{2}\right].$$

After some algebra with the y terms,

$$\frac{\mathrm{d}\sigma}{\mathrm{d}x\mathrm{d}y} = \left(\pi\right)\frac{4xf_1\left(x\right)e_a^2\,\alpha_{em}^2S}{Q^4}\left[\frac{1+\left(1-y\right)^2}{2}\right].$$

Finally,

$$\frac{\mathrm{d}\sigma}{\mathrm{d}x\mathrm{d}y} = \frac{\left(2\pi\right)xf_1\left(x\right)e_a^2\,\alpha_{em}^2S}{Q^4}\left[1+\left(1-y\right)^2\right].$$

This formula agrees with the well-known result for the unpolarized inclusive DIS cross-section.