Mathematical Physics
Research Group

Investigation of the Werner Basis Conjecture

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Abstract

Multiparticle states called Werner states are defined by their invariance under the same unitary evolution operator acting on each particle. Already demonstrating nonlocal quantum properties, these states also have potential application in quantum computation and cryptography. This project is an attempt to prove a conjecture which hopes to ensure the set of Werner states constructed from non-crossing polygon diagrams can uniquely represent all Werner states.

1. Introduction

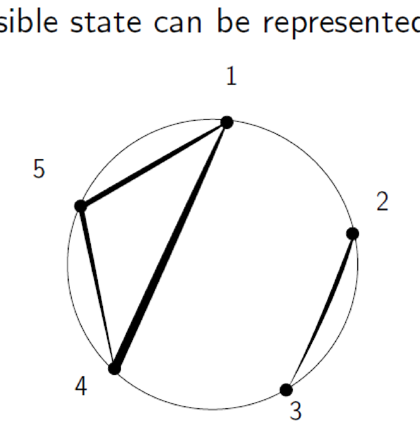
The discipline of quantum information science gained importance in the 1990s as researchers realized quantum states could be used for currently impossible computations. Many useful classes of states have been found, each with their individual uses, but the focus of this project is the class of Werner states. Werner states are defined by their invariance under the same unitary operation on every qubit position. This is important because many noise descriptions are based on unitary operators, so Werner states are useful in carrying information in noisy quantum channels [1], as it is possible to recover information lost to noise with known error-correcting operators.

2. Background

Quantum information is stored in qubits, the quantum analog to classical bits. A n-qubit quantum state lies in a complex vector space and is formed by taking linear combinations (or superpositions) of n-bit strings. For example, a 3-qubit state vector could be $\frac{1}{10}(|000\rangle - 5i\,|010\rangle + 8\,|101\rangle + \sqrt{10}\,|111\rangle)$. We use Dirac's ket notation ($|\psi\rangle$) to represent vectors (complex conjugate-transposed vectors are represented by bras ($\langle\psi|$)). Quantum states are conventionally normalized, which means $\langle\psi|\psi\rangle$ must $= 1$. In a 1-qubit system, the basis vectors are $|0\rangle = \begin{bmatrix}1\\0\end{bmatrix}$ and $|1\rangle = \begin{bmatrix}0\\1\end{bmatrix}$, combined to produce higher dimensional qubit spaces using a tensor product ($\otimes$), which denotes the Kronecker product of the two matrices, done by multiplying each entry in the first matrix by the second matrix, and forming a matrix of those values. For example, a basis vector for the 2-qubit space is $|0\rangle \otimes |1\rangle$, which is normally written as $|01\rangle$.

3. Forming Werner States

A diagram method has been developed to represent Werner States in n-qubits [1]. To construct these states, one begins with a non-crossing polygon diagram with a total of n points along the circle, not all of which must be connected to another. For example, in 5 qubits, a possible state can be represented by this diagram:



For each polygon in the diagram, one forms a sum of the cyclic permutations of the bit strings of the size of the polygon, with a normalization factor (so magnitude is 1) and roots of unity coefficients. Then, one turns this string into a density matrix. All the density matrices for each size bit string are summed, then normalizing the result. The final Werner state is formed by tensoring the density matrices in the positions specified by the points on the diagram.

$$C_{[1,4,5]}(001) = \tfrac{1}{\sqrt{3}}(|001\rangle + e^{\frac{2\pi i}{3}}\,|010\rangle + e^{\frac{4\pi i}{3}}\,|100\rangle)$$

$$C_{[2,3]}(01) = \tfrac{1}{\sqrt{2}}(|01\rangle - |10\rangle)$$

$$C_3 = \tfrac{1}{2}(C_{[1,4,5]}(001)C_{[1,4,5]}(001)^\dagger + C_{[1,4,5]}(011)C_{[1,4,5]}(011)^\dagger)$$

$$C_2 = C_{[2,3]}(01)C_{[2,3]}(01)^\dagger$$

$$\rho_{\mathcal{D}} = (C_3)_{[1,4,5]} \otimes (C_2)_{[2,3]}$$

Werner Basis Conjecture [1]

The states $\rho_{\mathcal{D}}$ (as $\mathcal{D}$ varies over all non-crossing polygon diagrams) form a basis for the space of Werner states (in the larger space of real linear combinations of Pauli tensors).

It has been shown that the size of this set matches the dimension of the Werner space, and the subject of our work was to prove this set is linearly independent.

4. Matrices and Independence

Most of our work was done with the assistance of the GAP system, a free and open software package. All code examples are written for this program. The standard test for independence, which begins supposing $\sum_{\mathcal{D}} c_{\mathcal{D}}\rho_{\mathcal{D}} = 0$, is only practical for 2 qubits, as the higher dimensions become computationally infeasible.
In 2 qubits, these are the two possible Werner States:

$$\begin{bmatrix}\frac14&0&0&0\\0&\frac14&0&0\\0&0&\frac14&0\\0&0&0&\frac14\end{bmatrix} \text{ and } \begin{bmatrix}0&0&0&0\\0&\frac12&-\frac12&0\\0&-\frac12&\frac12&0\\0&0&0&0\end{bmatrix}$$

This set is independent by observation.

5. Minor Tests

As constructed, the Werner diagram basis is not orthonormal, so we next tried to use GAP to orthonormalize each set. We were searching for a new non-zero value in some place in each successive matrix; however, the n-gon's matrix always only included values in places from all the preceding matrices' values. The next method we tried was centered on applying a summation function (basically the trace extended to superdiagonals) to the set of Werner Diagrams at each superdiagonal. Using the scalar nature of the summation function we could pull out the coefficients and sum each successive superdiagonal.

6. Hilbert-Schmidt Inner Product

After working with the summation function, we decided to create a matrix of the Hilbert-Schmidt Inner Product between each pair of diagrams, which would be square, and it can be shown that if the matrix is invertible, we would prove independence (therefore proving the Werner Basis Conjecture). In 3 and 4 qubits, the inner products were mostly real fractions. However, in higher dimensions, GAP began to give us results as combinations of roots of unity, which made calculating the determinant and eigenvalues difficult, even for GAP. For 6 and fewer qubits, GAP was capable of returning values, which led us to a conjecture on the determinant of these Hilbert-Schmidt Inner Product (HIP) matrices:

HIP Determinant Conjecture

$\det(A_n) = -\sum_{\ell=1}^{n-1} c_\ell \cdot (e^{\frac{2\pi i}{n}})^\ell$, such that $c_\ell = c_{n-\ell}$ and $c_\ell \neq 0$ and the sum $\neq 0$

Example of code used to create HIP matrices:

```
HIP := function(integer)
n:=integer;
c:=Catalan(n);
A:=IdentityMat(c);
B:=WD(n);
for i in [1..c] do;
for j in [1..c] do;
t:=TraceMat(B[i]*B[j]);
A[i][j]:=t; od; od; return A; end;
```

Example of output for HIP for 4 qubits (14 non-crossing polygon diagrams):

$$a := -\tfrac{1}{12}\cdot E(12)^4 - \tfrac{1}{24}\cdot E(12)^7 - \tfrac{1}{12}\cdot E(12)^8 + \tfrac{1}{24}\cdot E(12)^{11}$$

$$\begin{bmatrix}
\frac{1}{16}&\frac{1}{16}&\frac{1}{16}&\frac{1}{16}&\frac{1}{16}&\frac{1}{16}&\frac{1}{16}&\frac{1}{16}&\frac{1}{16}&\frac{1}{16}&\frac{1}{16}&\frac{1}{16}&\frac{1}{16}&\frac{1}{16}\\
\frac{1}{16}&\frac{1}{4}&\frac{1}{16}&\frac{1}{16}&\frac{1}{16}&\frac{1}{16}&\frac{1}{16}&\frac{1}{4}&\frac{1}{16}&\frac{1}{8}&\frac{1}{16}&\frac{1}{16}&\frac{1}{8}&\frac{1}{16}\\
\frac{1}{16}&\frac{1}{16}&\frac{1}{4}&\frac{1}{16}&\frac{1}{16}&\frac{1}{16}&\frac{1}{16}&\frac{1}{16}&\frac{1}{4}&\frac{1}{8}&\frac{1}{8}&\frac{1}{16}&\frac{1}{16}&\frac{1}{16}\\
\frac{1}{16}&\frac{1}{16}&\frac{1}{16}&\frac{1}{4}&\frac{1}{16}&\frac{1}{16}&\frac{1}{16}&\frac{1}{16}&\frac{1}{16}&\frac{1}{8}&\frac{1}{16}&\frac{1}{8}&\frac{1}{16}&\frac{1}{8}\\
\frac{1}{16}&\frac{1}{16}&\frac{1}{16}&\frac{1}{16}&\frac{1}{4}&\frac{1}{16}&\frac{1}{16}&\frac{1}{4}&\frac{1}{16}&\frac{1}{16}&\frac{1}{8}&\frac{1}{8}&\frac{1}{16}&\frac{1}{16}\\
\frac{1}{16}&\frac{1}{16}&\frac{1}{16}&\frac{1}{16}&\frac{1}{16}&\frac{1}{4}&\frac{1}{16}&\frac{1}{16}&\frac{1}{16}&\frac{1}{16}&\frac{1}{8}&\frac{1}{16}&\frac{1}{8}&\frac{1}{8}\\
\frac{1}{16}&\frac{1}{16}&\frac{1}{16}&\frac{1}{16}&\frac{1}{16}&\frac{1}{16}&\frac{1}{4}&\frac{1}{16}&\frac{1}{4}&\frac{1}{16}&\frac{1}{16}&\frac{1}{8}&\frac{1}{8}&\frac{1}{16}\\
\frac{1}{16}&\frac{1}{4}&\frac{1}{16}&\frac{1}{16}&\frac{1}{4}&\frac{1}{16}&\frac{1}{16}&1&\frac{1}{4}&\frac{1}{8}&\frac{1}{8}&\frac{1}{8}&\frac{1}{8}&0\\
\frac{1}{16}&\frac{1}{16}&\frac{1}{4}&\frac{1}{16}&\frac{1}{16}&\frac{1}{16}&\frac{1}{4}&\frac{1}{4}&1&\frac{1}{8}&\frac{1}{8}&\frac{1}{8}&\frac{1}{8}&0\\
\frac{1}{16}&\frac{1}{8}&\frac{1}{8}&\frac{1}{8}&\frac{1}{16}&\frac{1}{16}&\frac{1}{16}&\frac{1}{8}&\frac{1}{8}&\frac{1}{4}&\frac{1}{12}&\frac{1}{12}&\frac{1}{12}&a\\
\frac{1}{16}&\frac{1}{16}&\frac{1}{8}&\frac{1}{16}&\frac{1}{8}&\frac{1}{8}&\frac{1}{16}&\frac{1}{8}&\frac{1}{8}&\frac{1}{12}&\frac{1}{4}&\frac{1}{12}&\frac{1}{12}&a\\
\frac{1}{16}&\frac{1}{16}&\frac{1}{16}&\frac{1}{8}&\frac{1}{8}&\frac{1}{16}&\frac{1}{8}&\frac{1}{8}&\frac{1}{8}&\frac{1}{12}&\frac{1}{12}&\frac{1}{4}&\frac{1}{12}&a\\
\frac{1}{16}&\frac{1}{8}&\frac{1}{16}&\frac{1}{16}&\frac{1}{16}&\frac{1}{8}&\frac{1}{8}&\frac{1}{8}&\frac{1}{8}&\frac{1}{12}&\frac{1}{12}&\frac{1}{12}&\frac{1}{4}&a\\
\frac{1}{16}&\frac{1}{16}&\frac{1}{16}&\frac{1}{8}&\frac{1}{16}&\frac{1}{8}&\frac{1}{16}&0&0&a&a&a&a&\frac{1}{3}
\end{bmatrix}$$

Determinant of above matrix$=\frac{10935}{1125899906842624}$

7. Permutation Matrices

Another form of matrix can be constructed from non-crossing polygon diagrams, a special kind of permutation matrix (denoted by V). These are formed by creating a permutation matrix for each polygon, and tensoring these together to form the matrix for the diagram. As it is known the permutation matrices form a basis over the space of the diagram states [2], our next idea was to use the permutation matrices and their relation to the Werner Diagrams to prove independence. By definition of a basis, any Werner diagram can be written as a linear combination of permutation matrices, and it can easily be shown the top Werner Diagram can be written in the form

$$\rho_{n\text{-gon}} \propto \sum_{k=0}^{n-1} (e^{\frac{2\pi i}{n}})^k V_{n\text{-gon}}^k \tag{1}$$

We began exploring the expansion of Werner Diagrams and the powers of the top permutation matrices in the permutation basis, hoping to be able to express the powers of the top permutation matrix as a non-zero coefficient with the top matrix and some combination of lower diagrams, which we could then use induction on to prove the top permutation matrix was also a combination of Werner Diagrams. These combinations could then be used in (1) to show independence of the Werner diagrams.

Method based on Permutation Matrices

The set of Werner Diagrams is linearly independent if, when $\rho_{n\text{-gon}}$ is expanded in the permutation basis, the coefficient of $V_{n\text{-gon}}$ is nonzero.

Example of code for expanding a matrix in a basis

```
BasisCoef:=function(list,matrix)
A:=list; D:=[];
for i in [1..Length(A)] do;
Add(D,flatten(A[i])); od;
B:=matrix; C:=SolutionMat(D,flatten(B)); return C;
end;
```

Example of the expansion of powers of the $V_{4\text{-gon}}$

$$\begin{bmatrix}0&0&0&0&0&0&0&0&0&0&0&0&0&1\end{bmatrix} V_{4\text{-gon}}^1$$

$$\begin{bmatrix}1&-1&-1&0&-1&0&-1&1&1&1&1&1&1&-2\end{bmatrix} V_{4\text{-gon}}^2$$

$$\begin{bmatrix}-2&1&1&1&1&1&1&0&0&-1&-1&-1&-1&1\end{bmatrix} V_{4\text{-gon}}^3$$

N-gon coefficients

When expanding the powers of the top permutation matrices in the permutation basis, an interesting pattern developed. It began similar to an alternating Pascal's triangle, but by the fourth row ($n = 6$) the values began to differ. By the sixth row ($n = 8$), the sequence did not match any in the OEIS. Based on the symmetry and properties of the roots of unity, it can be shown the sum across any even row will always be real and across any odd row the sum will always be imaginary, however, we were unable to show the sum will not be zero.

Triangle of Coefficients of $V_{n\text{-gon}}$ to powers from 1 to $n-1$

$n=2$								1							
$n=3$							1		-1						
$n=4$						1		-2		1					
$n=5$					1		-3		3		-1				
$n=6$				1		-4		8		-4		1			
$n=7$			1		-5		17		-17		5		-1		
$n=8$	1		-6		33		-56		33		-6		1		

Based on our observations, we conjecture the expansion of the top permutation matrix squared is

$$V_{n\text{-gon}}^2 = (2-n)\cdot V_{n\text{-gon}} + \text{ others}$$

which we then would use for induction on higher powers of $V_{n\text{-gon}}$ to show the coefficient is nonzero.
We also observed the coefficients of the identity matrix in these expansions are the values from the triangle, but backwards across the rows.

Inverse Matrices

While trying to discover where these coefficients come from, we experimented with the expansion of the inverse of the top permutation matrices. The inverse of the k power of the $V_{n\text{-gon}}$ is the $n-k$ power, so we hoped understanding the power and its inverse's relationship would help us understand where the coefficients come from, as the power and the inverse have the same top matrix coefficient, sometimes with the opposite sign, when expanded in the permutation basis. Our observations led to this proven conjecture on the form of the inverse.

Expansion of $V_{n\text{-gon}}^{-1}$

$$V_{n\text{-gon}}^{-1} = \mathrm{Id} + \sum_{k=0}^{n}(-1)^k \sum_{|p|=k} V_{p\mathrm{Udots}}$$

8. Results and Further Work

While we were unable to prove the Werner Basis Conjecture, we were able to investigate several avenues for proof, and our latest idea on the expansion of Werner Diagrams in the permutation basis has the best chance of success. Our idea hinges on the understanding of the triangle of coefficients, as if we can understand where they come from and obtain a formula, we could show the coefficient for the top permutation matrix will never be zero, and our proof follows from there.

References

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