Investigation of the Werner Basis Conjecture

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- Some noise descriptions are based on unitary operators
- Useful in carrying information in noisy quantum channels (Lyons 2012)

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• Denotes the Kronecker product of the two matrices: done by multiplying each entry in the first matrix by the second matrix, and forming a matrix of those values

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- So, as above: $|0\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $|1\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, so

$$|0\rangle \otimes |1\rangle = \begin{bmatrix} 1 \cdot \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\ 0 \cdot \begin{bmatrix} 0 \\ 1 \end{bmatrix} \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

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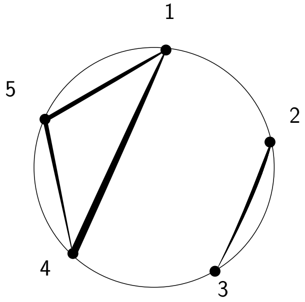
- Begin with a non-crossing polygon diagram with a total of *n* points along the circle
- In 5-qubit possible state:



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• Final matrix is 2^n by 2^n

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- Size of this set matches the dimension of the Werner space
- Subject of our work was to prove this set is linearly independent

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- 2 qubits: two possible Werner States:

$$\begin{bmatrix} \frac{1}{4} & 0 & 0 & 0 \\ 0 & \frac{1}{4} & 0 & 0 \\ 0 & 0 & \frac{1}{4} & 0 \\ 0 & 0 & 0 & \frac{1}{4} \end{bmatrix} \text{ and } \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & -\frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

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- Dependent Example:

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \text{and} \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix}$$

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- Difficulties: Worked for low n, however patterns were not clear and unable to generalize solutions to higher dimensions

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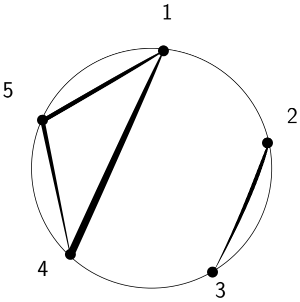

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- Vectors are made into columns for the permutation matrix, in original order

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- The permutation matrices form a basis over the space of the diagram states (Grassl 1998)
- Use the permutation matrices and relation to the Werner Diagrams to prove independence
- Any Werner diagram written as a linear combination of permutation matrices
- Can easily be shown the top Werner Diagram can be written in the form

$$\rho_{n\text{-gon}} \propto \sum_{k=0}^{n-1} (e^{\frac{2\pi i}{n}})^k V_{n\text{-gon}}^k \tag{1}$$

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Method based on Permutation Matrices

The set of Werner Diagrams is linearly independent if, when $\rho_{n\text{-gon}}$ is expanded in the permutation basis, the coefficient of $V_{n\text{-gon}}$ is nonzero.

- Example of code for expanding a matrix in a basis

```
BasisCoef:=function(list,matrix)
A:=list; D:=[];
for i in [1..Length(A)] do;
Add(D,flatten(A[i])); od;
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- Example of the expansion of powers of the $V_{\text{4-gon}}$

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} V^1_{\text{4-gon}}$$

$$\begin{bmatrix} 1 & -1 & -1 & 0 & -1 & 0 & -1 & 1 & 1 & 1 & 1 & 1 & 1 & -2 \end{bmatrix} V^2_{\text{4-gon}}$$

$$\begin{bmatrix} -2 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & -1 & -1 & -1 & -1 & 1 \end{bmatrix} V^3_{\text{4-gon}}$$

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- Symmetry and properties of the roots of unity: can be shown the sum across any even row will always be real and across any odd row the sum will always be imaginary
- Unable to show the sum will not be zero

Triangle of Coefficients of $V_{n\text{-gon}}$ to powers from 1 to $n-1$

$n=2$							1							
$n=3$						1		-1						
$n=4$					1		-2		1					
$n=5$				1		-3		3		-1				
$n=6$			1		-4		8		-4		1			
$n=7$		1		-5		17		-17		5		-1		
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- Conjecture the expansion of the top permutation matrix squared is

$$V_{n\text{-gon}}^2 = (2-n)\cdot V_{n\text{-gon}} + \text{ others}$$

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$n=8$		1		-6		33		-56		33		-6		1	

- Conjecture the expansion of the top permutation matrix squared is

$$V_{n\text{-gon}}^2 = (2-n) \cdot V_{n\text{-gon}} + \text{ others}$$

- Induction on higher powers of $V_{n\text{-gon}}$ to show the coefficient is nonzero

Triangle of Coefficients of $V_{n\text{-gon}}$ to powers from 1 to $n-1$

$n=2$								1						
$n=3$							1		-1					
$n=4$						1		-2		1				
$n=5$					1		-3		3		-1			
$n=6$				1		-4		8		-4		1		
$n=7$			1		-5		17		-17		5		-1	
$n=8$		1		-6		33		-56		33		-6		1

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- Observed the coefficients of the identity matrix in these expansions are the values from the triangle (backwards)

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- Power and the inverse have the same top matrix coefficient, sometimes with the opposite sign
- Observations led to this proven conjecture on the form of the inverse:

Expansion of $V_{n\text{-gon}}^{-1}$

$$V_{n\text{-gon}}^{-1} = \mathrm{Id} + \sum_{k=0}^{n} (-1)^k \sum_{|p|=k} V_{p \cup \mathrm{dots}}$$

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- Hinges on the understanding of the triangle of coefficients
- Understand where they come from and obtain a formula $->$ show the coefficient for the top permutation matrix will never be zero; our proof follows from there.

Thank you for your attention.

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